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The Rectangular Constant for Two-Dimensional Spaces

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1. INTRODUCTION

In [4], J. L. Joly defines the rectangular constant of a real normed linear space E, of dimension $\geqslant 2$, as the number

$$\mu(E) = \sup_{x \perp y} (\| x \| + \| y \|)/(\| x + y \|),$$

where $x \perp y$ means that $x \in E$ is orthogonal to $y \in E$ in the Birkhoff sense (B-orthogonal) [1, 2], i.e., $\| x \| \leqslant \| x + \lambda y \|$, $\forall \lambda \in \mathbb{R}$.

Joly proves the following properties of $\mu(E)$:

For every E, $2^{1/2} \leqslant \mu(E) \leqslant 3$.

If E is a prehilbert space, then $\mu(E) = 2^{1/2}$.

If $\mu(E) = 2^{1/2}$, then B-orthogonality is symmetric.

As a consequence of the above fact and of a well-known result of James [3], if $\mu(E) = 2^{1/2}$ and if E is of dimension $\geqslant 3$, then E is prehilbert space.

The purpose of this paper is to solve a problem posed by Joly in his work: "If E is a real normed linear space of dimension 2 and if $\mu(E) = 2^{1/2}$, then E is a prehilbert space."

From now on we shall suppose that E is a real normed linear space of dimension 2 (i.e., $\mathbb{R}^2$ with a norm), with an orientation ν, and we shall use the following notation:

$$S = \{x \in E: \| x \| = 1\},\ B = \{x \in E: \| x \| \leqslant 1\},$$

$$S_1 = \{p = x + y: x, y \in S,\ x \perp y,\ [x, y] = \nu\},$$

$$S_1' = \{p' = x - y: x, y \in S,\ x \perp y,\ [x, y] = \nu\}.$$

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The key result in [4] is that the area of the set B_1, internal to the curve S_1, is twice the area of the set B. To obtain this result, Joly gives a suitable parametrization of S_1, obtained from the fact that S is the boundary of a convex body and, therefore, it admits the following two parametrizations:

1. Let l be the length of S and ξ a point of S. Then, a possible parametrization of S is given by

$$\lambda \in [0, l) \to x(\lambda) = (x_1(\lambda), x_2(\lambda)) \in S,$$

where $x(\lambda)$ is the point of S whose distance to ξ, measured over S with the orientation ν, is equal to λ.

2. Given $\eta \in S$, the function

$$\theta \in [0, 2\pi) \to y(\theta) = (y_1(\theta), y_2(\theta)) \in S,$$

where $y(\theta)$ is the point of S whose angle with η, measured with the orientation ν, is equal to θ, is also a parametrization of S.

Then, if for a given $\xi \in S$, we take $\eta \in S$ as the first point such that ξ is B-orthogonal to η which we meet when S is described with the orientation ν, the following parametrization can be given for S_1:

$$\sigma = \lambda + \theta \in [0, 2\pi + l) \to p(\sigma) = x(\lambda) + y(\theta) \in S_1,$$

where $x(\lambda)$, $y(\theta)$ are such that $x(\lambda) \perp y(\theta)$, $[x(\lambda), y(\theta)] = \nu$.

The validity of the above result and the facts that S_1 is rectifiable and that $\mu(E) = 2^{1/2}$ implies $S_1 = 2^{1/2}S = S_1'$, has been discussed in [4].

Finally, Joly proves that $S_1 = 2^{1/2}S = S_1'$ implies the symmetry of B-orthogonality.

2. Previous Lemmas

To obtain a proof of Joly's conjecture we need the following results added to those of [4].

Lemma 1. *If* B-*orthogonality of E is symmetric, the areas of the parallelograms determined by any two vectors $x, y \in S$, such that $x \perp y$, are constant.*

Proof. The area of the parallelogram determined by $x = (x_1, x_2)$, $y = (y_1, y_2)$ is $|x_1 y_2 - x_2 y_1|$. Therefore, it suffices to prove that the function

$$A\colon \sigma \in [0, 2\pi + l) \to A(\sigma) = x_1(\lambda)\, y_2(\theta) - x_2(\lambda)\, y_1(\theta)$$

is constant.

Since the functions ($i = 1, 2$)

$$\sigma \in [0, 2\pi + l) \to x_i(\lambda), \qquad \sigma \in [0, 2\pi + l) \to y_i(\theta)$$

are of bounded variation, we only need to show

$$\int_{\sigma'}^{\sigma''} dA = \int_{\theta'}^{\theta''} \{x_1(\lambda)\, dy_2(\theta) - x_2(\lambda)\, dy_1(\theta)\} + \int_{\lambda'}^{\lambda''} \{y_2(\theta)\, dx_1(\lambda) - y_1(\theta)\, dx_2(\lambda)\} = 0$$

for every $\sigma' = \lambda' + \theta'$, $\sigma'' = \lambda'' + \theta''$, in $[0, 2\pi + l)$.

For this purpose, let $\{\sigma_0, \sigma_1, ..., \sigma_n\}$, with $\sigma_i = \lambda_i + \theta_i$, be any partition of $[\sigma', \sigma'']$. Since B-orthogonality of E is symmetric, $x(\lambda_i) \perp y(\theta_i)$ and $y(\theta_i) \perp x(\lambda_i)$. Hence, there exist $s_i \in [\lambda_{i-1}, \lambda_i]$, $t_i \in [\theta_{i-1}, \theta_i]$, such that

$$x_1(s_i)[y_2(\theta_i) - y_2(\theta_{i-1})] - x_2(s_i)][y_1(\theta_i) - y_1(\theta_{i-1})] = 0$$
$$y_2(t_i)[x_1(\lambda_i) - x_1(\lambda_{i-1})] - y_1(t_i)[x_2(\lambda_i) - x_2(\lambda_{i-1})] = 0$$

and the lemma follows.

LEMMA 2. *If $\mu(E) = 2^{1/2}$, then $x + y \perp x - y$ for every $x, y \in S$ such that $x \perp y$.*

Proof. Let $\mu(E) = 2^{1/2}$ and let $x, y \in S$, such that $x \perp y$. The inequality

$$\| x + y \| \leqslant \| x + y + t(x - y)\|, \qquad \forall t \in \mathbb{R},$$

easily follows from $\| x + y \| = 2^{1/2}$, which is a consequence of Joly's result $S_1 = 2^{1/2}S$, and from

$$((1 + t) + (1 - t))/\| x + y + t(x - y)\| \leqslant \mu(E) = 2^{1/2}.$$

LEMMA 3. *If $\mu(E) = 2^{1/2}$ and if $x', y', x'', y'' \in S$ are such that $x' \perp y'$, $x'' \perp y''$, $[x', y'] = [x'', y'']$, then*

$$A(x', x'') = A(y', y''),$$

where $A(x, y)$, with $x, y \in E \backslash \{0\}$, denotes the area of the set

$$\{ax + by: a, b \in \mathbb{R}^+, \| ax + by \| \leqslant 1\}.$$

Proof. Let $\sigma' = \lambda' + \theta'$, $\sigma'' = \lambda'' + \theta''$, such that $x' = x(\lambda')$, $y' = y(\theta')$, $x'' = x(\lambda'')$, $y'' = y(\theta'')$.

From the fact that $\mu(E) = 2^{1/2}$, it follows that $S_1 = 2^{1/2}S = S_1'$, and hence

$$
\begin{aligned}
2A(x' + y', x'' + y'')
&= \tfrac{1}{2}\int_{\sigma'}^{\sigma''} \{p_1(\sigma)\, dp_2(\sigma) - p_2(\sigma)\, dp_1(\sigma)\} \\
&= \tfrac{1}{2}\int_{\lambda'}^{\lambda''} \{x_1(\lambda)\, dx_2(\lambda) - x_2(\lambda)\, dx_1(\lambda)\} + \tfrac{1}{2}\int_{\theta'}^{\theta''} \{y_1(\theta)\, dy_2(\theta) - y_2(\theta)\, dy_1(\theta)\} \\
&\quad + \tfrac{1}{2}\int_{\theta'}^{\theta''} \{x_1(\lambda)\, dy_2(\theta) - x_2(\lambda)\, dy_1(\theta)\} + \tfrac{1}{2}\int_{\lambda'}^{\lambda''} \{y_1(\theta)\, dx_2(\lambda) - y_2(\theta) dx_1(\lambda)\}.
\end{aligned}
$$

As in the proof of Lemma 1, the two last addends are null. The two first are, respectively, $A(x', x'')$, $A(y', y'')$. Thus we have

$$2A(x' + y', x'' + y'') = A(x', x'') + A(y', y''). \tag{1}$$

Analogously, from the fact that $x' \perp -y'$, $x'' \perp -y''$, $[x', -y'] = [x'', -y'']$, it follows that

$$2A(x' - y', x'' - y'') = A(x', x'') + A(-y', -y'') = A(x', x'') + A(y', y''). \tag{2}$$

The above lemma implies $x' + y' \perp x' - y'$, $x'' + y'' \perp x'' - y''$. From this and the fact that $[x' + y', x' - y'] = [x'' + y'', x'' - y'']$, we obtain

$$2A(x', x'') = A(x' + y', x'' + y'') + A(x' - y', x'' - y''). \tag{3}$$

Finally, the lemma follows from (1), (2), (3).

3. Main Result

THEOREM. *If $\mu(E) = 2^{1/2}$, then E is a prehilbert space.*

Proof. Since affine transformations preserve B-orthogonality, we can suppose that $(1, 0), (0, 1) \in S$, $(1, 0) \perp (0, 1)$.

Let $x(\theta) = (x_1(\theta), x_2(\theta))$ be the point of S which forms with $(1, 0)$ the angle θ, measured in the positive sense, and let $y(\theta) = (y_1(\theta), y_2(\theta))$ be the *unique* (consequence of Lemma 3) point of S such that $x(\theta) \perp y(\theta)$, $[x(\theta), y(\theta)] = [(1, 0), (0, 1)]$.

If we prove that

$$\int_0^\theta x_1\, dx_1 + \int_0^\theta y_1\, dy_1 = 0, \qquad \int_0^\theta x_2\, dx_2 + \int_0^\theta y_2\, dy_2 = 0$$

for every $\theta \in [0, 2\pi)$, then

$$x_1^2(\theta) + y_1^2(\theta) = 1, \qquad x_2^2(\theta) + y_2^2(\theta) = 1.$$

Combining these equalities with (Lemma 1)

$$x_1(\theta)\, y_2(\theta) - x_2(\theta)\, y_1(\theta) = 1,$$

we find

$$[x_1(\theta) - y_2(\theta)]^2 + [x_2(\theta) + y_1(\theta)]^2 = 0,$$

from which it follows that *S is the circle* $x_1^2 + x_2^2 = 1$.

To prove that

$$\int_0^\theta x_k\, dx_k + \int_0^\theta y_k\, dy_k = 0 \qquad (k = 1, 2),$$

let

$$\sum_{i=1}^n \{x_k(t_i)[x_k(\theta_i) - x_k(\theta_{i-1})] + y_k(s_i)[y_k(\theta_i) - y_k(\theta_{i-1})]\}$$

be a sum of Riemann sums for such integrals. If we choose $s_i, t_i \in [\theta_{i-1}, \theta_i]$ so that

$$\begin{aligned} x(\theta_i) - x(\theta_{i-1}) &= \| x(\theta_i) - x(\theta_{i-1})\|\, y(s_i), \\ y(\theta_i) - y(\theta_{i-1}) &= -\| y(\theta_i) - y(\theta_{i-1})\|\, x(t_i), \end{aligned} \tag{4}$$

the absolute value of the above sum of Riemann sums is less than or equal to

$$\begin{aligned} &\sum_{i=1}^n \big| \| y(\theta_i) - y(\theta_{i-1})\| - \| x(\theta_i) - x(\theta_{i-1})\| \big| \\ &\qquad\qquad \times \frac{| x_k(\theta_i) - x_k(\theta_{i-1})|\, | y_k(\theta_i) - y_k(\theta_{i-1})|}{\| x(\theta_i) - x(\theta_{i-1})\|\, \| y(\theta_i) - y(\theta_{i-1})\|} \\ &\leqslant \sum_{i=1}^n \big| \| y(\theta_i) - y(\theta_{i-1})\| - \| x(\theta_i) - x(\theta_{i-1})\| \big|, \end{aligned} \tag{5}$$

since S is in the square with vertices $(\pm 1, \pm 1)$, and thus $\|(a, b)\| \geqslant \max\{a, b\}$.

Consequently, it remains to prove that the last sum is as small as we wish if we choose the partition $\{\theta_0, \theta_1, ..., \theta_n\}$ of $[0, \theta]$ sufficiently fine.

Lemma 3 tells us that

$$\int_{\theta'}^{\theta''} \{x_1\, dx_2 - x_2\, dx_1\} = \int_{\theta'}^{\theta''} \{y_1\, dy_2 - y_2\, dy_1\}$$

for every θ' and θ''. Let $\{\theta_0, \theta_1, ..., \theta_n\}$ be a partition of $[\theta', \theta'']$ and let

s_i, $t_i \in [\theta_{i-1}, \theta_i]$ be as in (4). The corresponding Riemann sum of the first integral is

$$\sum_{i=1}^{n} \{x_1(s_i)[x_2(\theta_i) - x_2(\theta_{i-1})] - x_2(s_i)[x_1(\theta_i) - x_1(\theta_{i-1})]\}$$

$$= \sum_{i=1}^{n} \| x(\theta_i) - x(\theta_{i-1})\| \, [x_1(s_i)\, y_2(s_i) - x_2(s_i)\, y_1(s_i)]$$

$$= \sum_{i=1}^{n} \| x(\theta_i) - x(\theta_{i-1})\|.$$

Analogously, the Riemann sum of the second integral is

$$\sum_{i=1}^{n} \| y(\theta_i) - y(\theta_{i-1})\|.$$

Therefore, for every $\epsilon > 0$ there exists a partition P_ϵ of $[\theta', \theta'']$ such that, for every $P = \{\theta_0, \theta_1, ..., \theta_n\}$ finer than P_ϵ,

$$\left| \sum_{i=1}^{n} D_i \right| < \epsilon, \tag{6}$$

where

$$D_i = \| x(\theta_i) - x(\theta_{i-1})\| - \| y(\theta_i) - y(\theta_{i-1})\|.$$

To show the sum in (5) can be made small, suppose P_ϵ is a partition of $[0, \theta]$ chosen so that, if $P = \{\theta_0, \theta_1, ..., \theta_n\}$ is finer than P_ϵ,

$$\left| \sum_{i=1}^{n} D_i \right| = \left| \sum_{P_+} D_i + \sum_{P_-} D_i \right| < \epsilon,$$

where $P_+(P_-)$ denotes the set of addends D_i which are $\geqslant 0$ (<0).

By virtue of (6) we can refine $P_-(P_+)$ so that the sum of D_i's relative to it is less than ϵ. Consequently,

$$\sum_{P_+} D_i < 2\epsilon, \qquad \left(\sum_{P_-} D_i < 2\epsilon\right),$$

and therefore

$$\sum_{i=1}^{n} | D_i | = \sum_{P_+} D_i - \sum_{P_-} D_i < 4\epsilon$$

for every P finer than P_ϵ.

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